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A GENERAL THEORY ON MULTIVARIABLE ERROR-FREE RIPPLE-FREE SAMPLED-DATA CONTROL SYSTEMS

by

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ABSTRACT

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This report considers a multivariable system, described by the P-canonical structure, utilizing two digital controllers, one in the forward loop and one in the feedback loop. The controllers are designed to obtain the fastest error-free and ripple-free input-output responses, and the fastest neutralization of disturbances. Two matrices are fundamental to the design, the matrix K, relating the output response and the input response, and the matrix L, relating the output due to a disturbance in the system forward loop to the disturbance itself. The performance criteria, the fundamental requirements for interacting or non-interacting systems and the conditions for independent output restoration are implemented directly into these two matrices. The matrices are restricted by the nature of the inputs, the disturbances and by the plant itself. The advantages of the proposed design method are as follows: the chosen configuration of the digital controllers used in the structure does not require that extra restrictions be placed on the K matrix if the plant is unstable; the system is designed as an n variable system and does not use n single variable systems constructed artificially from the original multivariable system, while retaining a relatively simple design procedure.

Author

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INTRODUCTION

A particular method for the optimal design of digital controllers for a large class of multivariable systems has been chosen and developed. The method chosen is based extensively on a paper by Hung[5], on single variable systems. Hung utilized two digital controllers, one in the forward loop and one in the feedback loop of a sampled data control system. There are two distinct advantages in this controller configuration. By using two controllers the desired input-output response could be obtained independently of disturbances acting on the system (anywhere other than at the input). By having one controller in the feedback loop the input-output response was not affected by unstable plant. There has been some fairly recent work on the design of controllers for multivariable systems[6,7], but compared with that on single variable systems it can hardly be considered extensive. The method resulting from this paper, obtained from extending the single variable method to multivariable systems, has several factors to commend it. These are itemized and discussed briefly.

1) General system criteria, which will be introduced later, such as:

independent output restoration,
interacting system,
non-interacting system,

are implemented as the first step in the design. They determine whether the two basic transmission matrices are diagonal or non-diagonal. These two matrices, K and L, will be introduced more fully later.

2) The system itself places certain essential restrictions on these matrices. Obtaining these restrictions is the essence of the bulk of this paper, and as long as these restrictions are included K and L may be otherwise chosen as desired. If one controller is placed in the feedback loop then the input-output matrix, K, is unrestricted by unstable plant.

3) The criteria determining the output of the system, Minimal Prototype or Ripple Free, and elimination of the disturbance effect, are implemented directly in the basic matrices K and L.

Thus the method of designing in terms of the two basic matrices ties together the general system criteria, and involves at the same time the restrictions essential to obtain the desired type of output response.

4) The resulting method developed in the paper permits a "true" multivariable design: the system does not have to be broken down into an equivalent number of single variable systems[7], with its consequent divorce from the actual multivariable problem.

For a general multivariable system, with n inputs and n outputs, matrix notation may be used. Capital letters denote a matrix and lower-case letters the elements of a matrix, for example

$R = R(z)$ is the input matrix.

$r_j = r_j(z)$ is the j^{th} element of the input matrix R.

Referring to Fig. 1,

$$\text{Input, } R = \left\| \begin{matrix} r_1 \\ r_2 \\ \vdots \\ r_j \\ \vdots \\ r_n \end{matrix} \right\| \qquad \text{Output, } C = \left\| \begin{matrix} c_1 \\ c_2 \\ \vdots \\ c_j \\ \vdots \\ c_n \end{matrix} \right\|$$

where C(z) is the output due to R(z) acting independently of the disturbance.

Similarly,

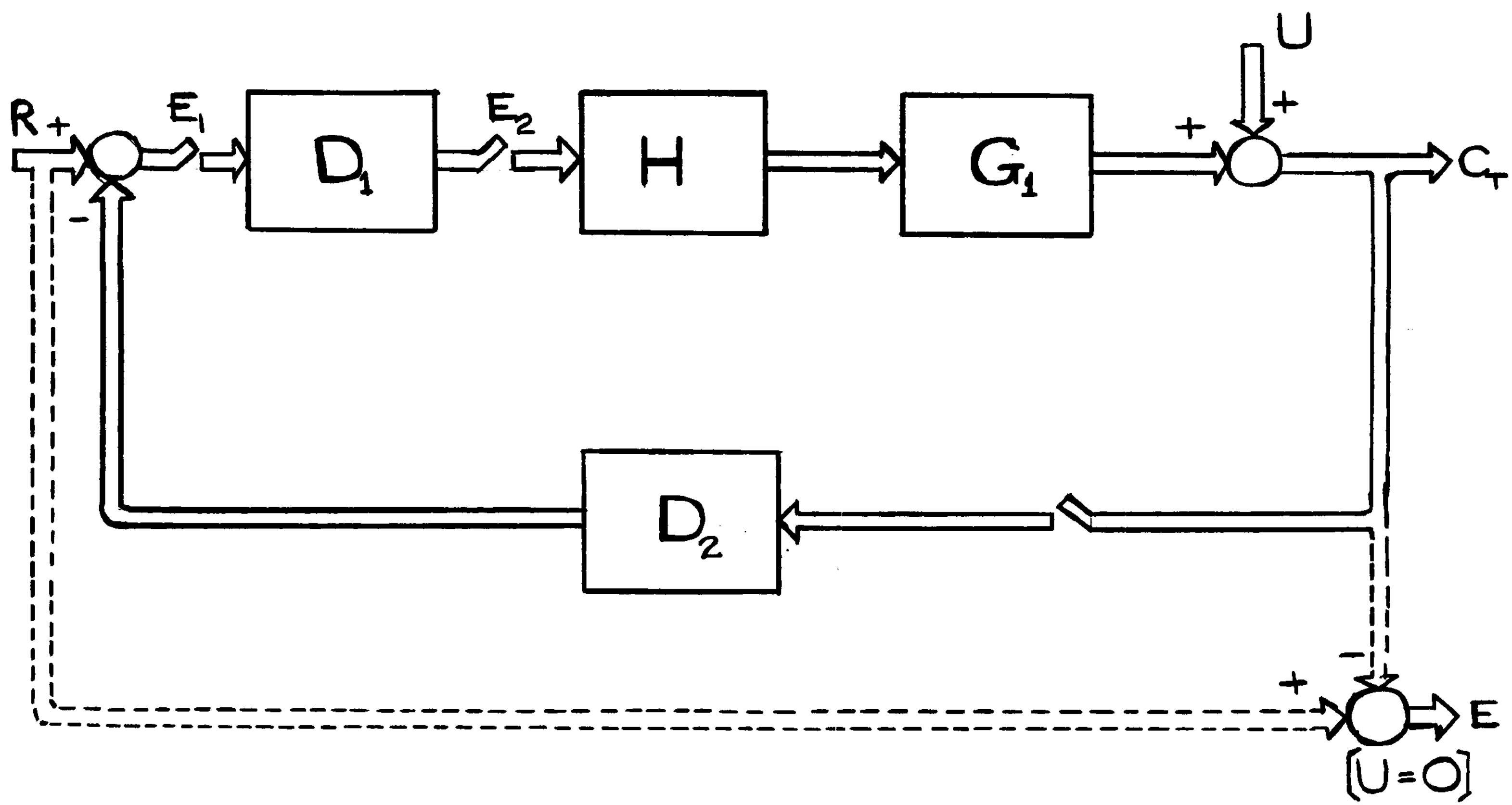

Figure 1. The system configuration

$$U = \begin{Vmatrix} u_1 \\ u_2 \\ \vdots \\ u_j \\ \vdots \\ u_n \end{Vmatrix} \qquad C' = \begin{Vmatrix} c' \\ c'_2 \\ \vdots \\ c'_j \\ \vdots \\ c'_n \end{Vmatrix}$$

where C' is the output due to the disturbance, U acting alone.

If there is a relationship between C and R of the form,

$$C = KR \tag{1}$$

where K is an n by n matrix, then the system may be structurally classified as 'P-canonical[4]. A large class of systems may be represented by this structure and this paper is concerned only with this structure.

A very desirable practical feature of multivariable systems has been named by Freeman[6] as "independent output restoration". A system is said to have independent output restoration if changes in the individual outputs of the system do not affect the other outputs. The P-canonical structure conveniently describes such systems. Thus restricting the structure of the system to the P-canonical is, for many practical cases, not a disadvantage.

From Equation (1),

$$c_1 = k_{11}r_1 + k_{12}r_2 + \cdots + k_{1p}r_p + \cdots + k_{1n}r_n$$

and for the j^{th} output,

$$c_j = k_{j1}r_1 + \cdots + k_{jp}r_p + \cdots + k_{jn}r_n \tag{2}$$

where k_{jp} is the element of K, the input transmission matrix, belonging to the j^{th} row and the p^{th} column.

From Equation (2) it is clear that, in general, each output is dependent upon all of the inputs, and when any one output, say c_j is dependent upon more than one input the system is termed "interacting". When c_j is dependent only upon r_j (for all j) the system is termed "non-interacting". For a non-interacting system it is clear that K is a diagonal matrix.

Just as C has been related to R by the input transmission matrix K, C' may be related to U by an n by n matrix L, the disturbance transmission matrix, so that,

$$C' = LU. \tag{3}$$

It will now be shown that for independent output restoration L must be a diagonal matrix. Suppose L is not a diagonal matrix, then in general,

$$c'_j = l_{j1}u_1 + l_{j2}u_2 + \cdots + l_{jp}u_p + \cdots + l_{jn}u_n \tag{4}$$

where l_{jp} is the element of L corresponding to the j^{th} row and the p^{th} column of L. It is evident from Eq. (4) that a disturbance on one output will affect all the other outputs. This is contrary to the desirable feature of independent output restoration. Thus so that a change of one output does not affect any of the other outputs it is clear that L <u>must be a diagonal matrix.</u>

The design criteria used in this paper are an integral part of the design theory and will in fact be used directly and quantitatively. Two criteria are used in describing two different general input responses, namely, Minimal Prototype Response, and Ripple Free Response. The minimal prototype response requires that the output reach the desired value in the least possible number of sampling instants. The ripple free response requires that steady-state outputs contain no inter-sampling ripple. In general the number of sampling instants taken to reach the final zero error is greater for the ripple free system than for the minimal prototype system response. It is also required that the disturbance effect be neutralized as soon as possible after it appears, and therefore C' must go to zero, at the sampling instants for a minimal prototype response, or without any ripple for a ripple-free system response.

BASIC EQUATIONS

Referring to Fig. 1, the basic equations will now be developed. Matrix notation is used throughout: E, E_1, E_2, , R, C and C' are column matrices, representing n-vectors, D_1, D_2 and G and H are n by n matrices. 0 is the zero matrix, I the identity matrix.

Since the system is assumed linear, superposition holds true and C and C' may be treated separately and added together to give the total output. The following equations are derived in Appendix III: primed variables are used when the disturbance U is considered alone, so that with U = 0,

$$C = \left[I + GD_1D_2\right]^{-1} GD_1R \quad . \tag{5}$$

With R = 0,

$$C' = \left[I + GD_1D_2\right]^{-1} U \quad . \tag{6}$$

From Equations (1) and (3), K and L may be re-defined as:

$$K = \left[I + GD_1D_2\right]^{-1} GD_1 \tag{7}$$

$$L = \left[I + GD_1D_2\right]^{-1} \tag{8}$$

and substituting (8) into (7),

$$K = LGD_1 \quad . \tag{9}$$

The following equations can also be shown:

$$E_2 = G^{-1}KR \tag{10}$$

$$L = I - KD_2 \quad . \tag{11}$$

For independent output restoration L must be a diagonal matrix and from Equation (8), GD_1D_2 must itself be a diagonal matrix. This is Freeman's[6] condition for independent output restoration. Finally:

$$E'_2 = -G^{-1}KD_2U \tag{12}$$

$$E = \left[I - K\right] R. \tag{13}$$

The idea lying behind the design method will now be given. The end result of the design method are the optimal controller matrices D_1 and D_2 that will give the desired response types. D_1 and D_2 are defined by K, L and G, Equations (11) and (9). G is assumed to be fixed plant. By obtaining K and L therefore, D_1 and D_2 may be obtained.

K and L however, cannot be chosen arbitrarily, but must contain certain factors, factors which are necessary for stability considerations and the obvious need for having controllers which are physically realizable. Various restrictions must be placed on K and L therefore, and these restrictions determine the response. This response is the optimum for the structure and the particular G matrix. The remainder of this paper is concerned with the determination of these restrictions and the practical procedure for determining D_1 and D_2 from the resulting K and L matrices. Restrictions for four variations are developed. Firstly the simpler non-interacting case, for both minimal prototype and the extension to ripple-free design. Secondly the interacting case is treated, again for minimal prototype and ripple-free designs.

RESTRICTIONS ON SYSTEM TRANSFER FUNCTION

NON-INTERACTING SYSTEMS

It has already been stated that K is diagonal. For independent output restoration, L and therefore GD_1D_2 are diagonal, and from Equation (11) it can be seen that KD_2 is diagonal. Since KD_2 and K are both diagonal, for the non-interacting system, D_2 is also diagonal. Now since GD_1D_2 is a diagonal matrix and D_2 is diagonal then GD_1 must also be diagonal.

The minimal prototype restrictions will now be derived. From Appendix I,

$$e_j = (1 - k_{jj})r_j \tag{14}$$

where k_{jj}, e_j and r_j are the j^{th} elements in their respective matrices. It is assumed that the inputs are deterministically describable by functions of the form:

$$r_j = A_j/(1-z^{-1})^{m_j} \text{ for all j,} \tag{15}$$

where A_j is a finite polynomial in z^{-1}. Then, if e_j is to go to zero in the steady-state, the denominator of r_j must be contained as zeros in $(1 - k_{jj})$, for all j, from Equation (14), or

$$(1 - k_{jj}) = (1 - z^{-1})^{m_j} F_j \tag{16}$$

where for all j, F_j is a finite polynomial in z^{-1}. This restriction on the elements of the diagonal K matrix is equivalent to

$$\begin{aligned} &k_{jj}(1) = 1 \\ &k'_{jj}(1) = 0 \\ &\vdots \\ &k_{jj}^{m_j-1}(1) = 0 \end{aligned} \tag{17}$$

where k_{jj} is differentiated m_j - 1 times with respect to z^{-1}, for all j. This is condition 1.

The mathematical form of the G matrix elements is in general:

$$g_{ij} = \frac{z^{-t_{ij}}\left(p_{ij}^{o} + p_{ij}^{1} z^{-1} + \cdots\cdots + p_{ij}^{a_{ij}} z^{-a_{ij}}\right)}{q_{ij}^{o} + q_{ij}^{1} z^{-1} + \cdots\cdots\cdots + q_{ij}^{b_{ij}} z^{-b_{ij}}} \tag{18}$$

for the element in the i^{th} row and j^{th} column. The term q_{ij}^{o} must always be present and also for practical plants t_{ij} is at least unity with q_{ij}^{o} present.[1,2,3] A term such as $z^{-t_{ij}}$ is called the " plant transport lag " of the i, j^{th} element.

From Equation (9)

$K = LGD_1$

or

$$D_1 = G^{-1}L^{-1}K. \tag{19}$$

Since GD_1 is diagonal, and G is in general non-diagonal, D_1 must be non-diagonal. Equation (19) is treated element-wise in Appendix I and a general expression obtained:

$$d_{ij}^{1} = g_{ij}^{-1} k_{jj}/l_{jj} \tag{20}$$

where g_{ij}^{-1} is the i, j^{th} element of G^{-1}, the inverse of G; d_{ij}^{1} is the element of D_1 corresponding to the i^{th} row and j^{th} column. From $L = I - KD_2$, l_{jj} will always contain a constant term in its polynomial in z^{-1} and so unless the transport lag of $1/g_{ij}^{-1}$ is contained in k_{jj}, then from (20), it will appear directly in the denominator of d_{ij}^{1} and would therefore give a physically unrealizable element d_{ij}^{1}. Considering Equation (20) again, the second restriction on k_{jj} is then evident. For all j, k_{jj} must contain the maximum transport lag of $1/g_{ij}^{-1}$ as i = 1... n, as a factor in its numerator. This is condition 2.

Substituting Equation (9) into Equation (10), and since L and GD_1 are diagonal,

$$E_2 = D_1 LR \tag{21}$$

For the elements of the column matrix E_2, in general, from Appendix I,

$$e_j^2 = \sum_{p=1}^{n} l_{pp} d^1_{jp} r_p \tag{22}$$

where e_j^2 is the j^{th} element of the matrix E_2. Also from (12),

$$E'_2 = -G^{-1} K D_2 U$$

and from Appendix I

$$e_j^{2'} = - \sum_{i=1}^{n} g^{-1}_{ji} \; k_{ii} d^2_{ii} \, u_i \tag{23}$$

where d^2_{ii} is the ii^{th} element of the matrix D_2.

Now consider Equation (20),

$$d^1_{ij} = g^{-1}_{ij} k_{jj} / l_{jj} \; .$$

Suppose g^{-1}_{ij} contains unstable poles, and suppose that they are not cancelled by zeros of k_{jj}, then d^1_{ij} will contain as its poles the poles of g^{-1}_{ij}. Equation (20) gives

$$k_{jj} = l_{jj} d^1_{ij} / g^{-1}_{ij} \; . \tag{24}$$

For the present, any outside zeros of g^{-1}_{ij}, which from Equation (24) would cause k_{jj} to be unstable, will be put to one side and dealt with later. The presence of the unstable poles of g^{-1}_{ij} as poles of d^1_{ij} is, however, effecting a pole-zero cancellation. Pole-zero cancellations interfering with the outside poles of any transfer function in a system, always produce instability. In this case, if d^1_{ij} contains unstable poles, not including those which are outside zeros of l_{jj}, for i = 1. . . n, then it is clear from Eq. (22) that e^2_j will be unstable, although Equation (24) would not indicate any instability of k_{jj}.

Equation (23) also shows that unless the outside poles of g_{ij}^{-1} are cancelled by zeros of k_{ii} or d_{ii}^{2}, then $e_j^{2'}$ will also be unstable.

Then it is evident that to give a stable controller element d_{ij}^{1} and to ensure that $e_j^{2'}$ is stable, any unstable poles of g_{ij}^{-1} must be contained in k_{jj}. This is the third restriction on k_{jj}.

For $j = 1 \ldots n$, k_{jj} must contain as its zeros the outside poles of g_{ij}^{-1} for $i = 1 \ldots n$. These outside poles of g_{ij}^{-1} are in fact the zeros of the determinant of G. This is condition 3.

Returning now to the case where g_{ij}^{-1} has outside zeros and referring again to Equation (20),

$$d_{ij}^{1} = g_{ij}^{-1} \, k_{jj}/l_{jj}$$

it is clear that these outside zeros of g_{ij}^{-1} cannot be contained as poles of k_{jj}, as an unstable k_{jj} would of course entail an unstable c_j. The outside zeros of g_{ij}^{-1} cannot be made outside zeros of d_{ij}^{1}, as this would mean pole-zero cancellation again. There is no reason, of course, why outside zeros of k_{jj} cannot be made outside zeros of d_{ij}^{1}. The only way to prevent pole-zero cancellation between the outside zeros of g_{ij}^{-1} and d_{ij}^{1} is to cancell the outside zeros of g_{ij}^{-1} by zeros of l_{jj}. Thus the fourth restriction may be formulated. For all j any outside zeros of g_{ij}^{-1} as $i = 1 \ldots n$ must be contained as the zeros of l_{jj}. These outside zeros of G^{-1} are related to the outside poles of G. This is condition 4a.

It is reasonable to assume that U is deterministically describable by a set of elements,

$$u_j = B_j/(1 - z^{-1})^{n_j} \tag{25}$$

where B_j is a polynomial in z^{-1} of finite length. From Equation (4)

$$c'_j = l_{jj} u_j \quad . \tag{26}$$

For the disturbance effect to be zero as soon as possible it can be seen from Equation (26) that l_{jj} should be of the form:

$$l_{jj} = (1 - z^{-1})^{n_j} P_j \tag{27}$$

where P_j is a finite polynomial in z^{-1}. This is condition 4b. Restrictions 4a and 4b may be conveniently condensed into one set of similar conditions. For all j,

$$
\begin{aligned}
&l_{jj}(1) = 0 \\
&l'_{jj}(1) = 0 \\
&\vdots \\
&l_{jj}^{n_j - 1}(1) = 0 \\
&l_{jj}\ (\text{first outside zero of } g_{ij}^{-1} \text{ as } i = 1 \ldots n) = 0 \\
&\vdots \\
&l_{jj}\ (\text{last outside zero of } g_{ij}^{-1} \text{ as } i = 1 \ldots n) = 0\ .
\end{aligned}
\qquad (28)
$$

This may be termed condition 4. If one or more of the outside zeros is raised to the q^{th} power then l_{jj} must be differentiated $(q - 1)$ times, and $l_{jj}, l'_{jj} \ldots l_{jj}^{q-1}$ at this zero equated to numeric zero. The example will illustrate this point.

These constitute the restrictions for the non-interacting, minimal prototype response.

Additional Restrictions for Ripple Free Response

The requirements for ripple free outputs is that the plant inputs be smooth, taking the form of a step function, a ramp, etc. The inputs to the plant are the hold outputs. To give the desired smooth outputs, the holds must conform to an elementary restriction and the hold inputs must be smooth themselves.

Suppose h_j is the order of the jj^{th} hold in H, which is diagonal, then

$$h_j + 1 \geq \phi_j \qquad (29)$$

where ϕ_j is the order of the hold input e_j^2. This can be seen directly from the mechanics of the hold operation. Thus, e_j^2 must be of the form,

$$e_j^2 = Q_j/(1 - z^{-1})^{\phi_j} \tag{30}$$

where Q_j is a polynomial in z^{-1}, of finite length so that a smooth hold input is eventually reached. If f_{ji} is the number of $(1 - z^{-1})$ factors in the numerator of g_{ji}^{-1}, then from Equation (10) and the working in Appendix I,

$$e_j^2 = \sum_{i=1}^{n} g_{ji}^{-1} k_{ii} r_i \tag{31}$$

it can be seen that the order of the j^{th} hold, h_j, must comply with the following restriction: For all j,

$$h_j + 1 \geq m_i - f_{ji} \qquad \text{as } i = 1 \ldots n \tag{32}$$

This is condition 5a. m_i is the order of the i^{th} input, r_i.

Since the input to the hold must itself be smooth to give a smooth output, then from Equation (31), $g_{ji}^{-1} k_{ii}$ must be a finite polynomial in z^{-1} and so a further restriction on k_{ii} is necessary. For all j, k_{jj} must contain as its zeros all the poles of g_{ij}^{-1} for $i = 1 \ldots n$. This is condition 6a.

The ripple-free disturbance function is treated in a similar way. From Equation (23)

$$e_j^{2'} = -\sum_{i=1}^{n} g_{ji}^{-1} k_{ii} d_{ii}^2 u_i \tag{33}$$

and assuming $e_j^{2'}$ is of the form,

$$e_j^{2'} = Q_j' /(1 - z^{-1})^{\phi_j'} \tag{34}$$

where Q_j' is a finite polynomial by the same reasoning as before; Q_j and Q_j' do not contain any $(1 - z^{-1})$ factors. Then again,

$$h_j + 1 \geq \phi_j' \tag{35}$$

and this gives a further restriction on the hold order, for all j,

$$h_j + 1 \geq n_j - f_{ji} \quad \text{as } i = 1 \quad n, \tag{36}$$

where n_i is the order of the i^{th} disturbance. This is condition 5b.

Now $k_{ii}d_{ii}^2$ is certainly finite itself, consider Equation (11), $L = I - KD_2$, where the L matrix is compared of finite elements, so that to obtain the required smooth hold input, $e_j^{2'}$; for all j, k_{jj} must contain as its zeros all the poles of g_{ij}^{-1}, for i = 1. . . n. This is the same as the input response condition 6a.

Summarizing the additional restrictions 5a and 5b, for all j:

$$h_j + 1 \geq (m_i \text{ and } n_i) - f_{ji} \quad \text{for } i = 1 \dots n \tag{37}$$

where f_{ji} is the number of $(1 - z^{-1})$ factors in the numerator of g_{ji}^{-1}, therefore condition 5.

k_{jj} must contain as its zeros all the poles of g_{ij}^{-1} as i = 1. . . n, for all j and so, condition 6, restating 6a for convenience here.

This concludes the restriction theory for the non-interacting case.

INTERACTING SYSTEMS

Considering Equation (2), for an interacting system the K matrix is non-diagonal. To retain independent output restoration L is still a diagonal matrix and GD_1D_2 must also be diagonal. Equation (9),

$$K = LGD_1$$

shows that GD_1 must be a non-diagonal matrix, since K is non-diagonal and L is diagonal. Consider now Equation (11),

$$L = I - KD_2$$

then KD_2 is a diagonal matrix and it is also clear from this that D_2 must be a non-diagonal matrix.

As before, this theory deduces the necessary restrictions on K and L and will enable D_1 and D_2 to be designed for the fastest reduction of the disturbance to zero, and the fastest response time for minimal prototype or ripple free systems. The K matrix reached after the restrictions have been applied is the "minimum" K matrix. The elements of K may be made longer to suit the design problem, all that the restriction theory requires is that the elements of K, and also those of L, contain certain specified factors.

The restrictions for the minimal prototype response will be developed first. For U = 0, Equation (13) states:

$$E = \left[I - K \right] R$$

and according to the result reached in Appendix I,

$$e_j = (1 - k_{jj})\, r_j - \sum_{\substack{i=1 \\ i \neq j}}^{n} k_{ji} r_i \tag{38}$$

then by the same reasoning used to derive condition 1, for the error e_j to settle to zero, in general: For all j, $(1 - k_{jj})$ must contain the factor $(1 - z^{-1})^{m_j}$, and k_{ij} must contain the factor $(1 - z^{-1})^{m_j}$ for all j as $i = 1..n$, $i \neq j$. This restriction is equivalent to: For all j,

$$\begin{aligned} k_{jj}(1) &= 1 \\ k'_{jj}(1) &= 0 \\ &\vdots \\ k_{jj}^{m_j - 1}(1) &= 0 \end{aligned} \tag{39}$$

and for all j as $i = 1..n$, $i \neq j$,

$$\begin{aligned} k_{ij}(1) &= 0 \\ k'_{ij}(1) &= 0 \\ &\vdots \\ k_{ij}^{m_j - 1}(1) &= 0 \end{aligned} \tag{40}$$

where k_{jj}, k_{ij} are differentiated with respect to z^{-1}. This is condition 7.

From Equation (19) and the matrix manipulation in Appendix I,

$$d^1_{ij} = \sum_{p=1}^{n} g^{-1}_{ip} \, k_{pj}/l_{pp} \tag{41}$$

and from Equation (41), by reasoning exactly similar to that used in deriving condition 2, it may be stipulated that: To ensure d^1_{ij} is physically realizable, k_{pj} must contain the maximum transport lag of $1/g^{-1}_{ip}$ as $i = 1..n$, for all p, j. This is condition 8.

Again from Equation (41) and by the reasoning used to derive condition 3, then to prevent d^1_{ij} from containing as its poles the outside poles of

$$\sum_{p=1}^{n} g^{-1}_{ip}$$

then, for all p, j, k_{pj} must contain as its zeros the outside poles of g^{-1}_{ip} as $i = 1...n$. These outside poles are the zeros of the determinant of G. This is condition 9.

Considering Equation (26), which applies in this interacting case as well as the non-interacting case, then by reasoning as in condition 4b, the following set of restrictions must be applied to the matrix L. For all j,

$$\begin{aligned} l_{jj}(1) &= 0 \\ l'_{jj}(1) &= 0 \\ &\vdots \\ l^{n_j - 1}_{jj}(1) &= 0 \end{aligned} \tag{42}$$

where the elements are differentiated with respect to z^{-1}. This is condition 10.

Consider now condition 4a of the non-interacting case, and at the same time consider Equation (41). There is no restriction on l_{jj} which is analogous to condition 4a. The outside zeros of g_{ip}^{-1} as $p = 1 \ldots n$ need not be made the outside zeros of l_{pp}, as the summation performed in Equation (41) does not result in the direct, and therefore harmful, appearance of the outside zeros of g_{ip}^{-1} for all p.

These constitute the restrictions on the K and L matrices for the interacting minimal prototype system.

Additional Restrictions for Ripple-Free Systems

Consider Equation (10),

$$E_2 = G^{-1}KR \;.$$

From Appendix I,

$$e_j^2 = g_{j1}^{-1}(k_{11}r_1 + k_{12}r_2 + \cdots + k_{1n}r_n) + \cdots + g_{jp}^{-1}(k_{p1}r_1 + \cdots + k_{pn}r_n) + \cdots + g_{jn}^{-1}(k_{n1}r_1 + \cdots + k_{nn}r_n) \;. \tag{43}$$

By reasoning developed for condition 6a, for smooth hold inputs then from Equation (43) the following condition is needed: All the poles of g_{ij}^{-1} for $i = 1 \ldots n$ must be contained as zeros of k_{jp} as $p = 1 \ldots n$. This is condition 11.

Considering Equation (12),

$$E_2' = -G^{-1}KD_2U$$

then if condition 11 has already been applied, the hold input E_2' is smooth itself. This is readily seen by comparison of Equations (10) and (12) and remembering that KD_2 is a matrix of finite elements.

Equation (43) may be compared with Equation (31), and then if the same reasoning be applied for this interacting case as was applied to the non-interacting case of condition 5a, it will be evident that a more stringent restriction must be placed on the order of the hold elements:

For all i,

$$h_i + 1 \geq m - f_i' \tag{44}$$

where f_i' is the least number of $(1 - z^{-1})$ factors in the numerator of g_{ij}^{-1} for all j, and where m is the maximum order of all the inputs. This is condition 12a.

In an exactly analogous manner, for $e_j^{2'}$, a similar restriction is also required: For all i,

$$h_i + 1 \geq n - f_j' \tag{45}$$

where n is the maximum order of the disturbance elements, u_j for all j. This is condition 12b.

Conditions 12a and 12b can be conveniently summarized into one composite restriction: For all i,

$$h_i + 1 \geq (m,n - f_j') \ . \tag{46}$$

This is condition 12.

This concludes the theory of the interacting system restrictions.

DESIGN PROCEDURES

In the preceding theory restrictions have been developed that are to be applied to the hold elements and to the K and L matrix elements. The design procedure for the non-interacting case is simpler than that used in the interacting case. Examples of each design method are given later. Both methods of design rely upon Equations (11) and (19),

$$L = I - KD_2$$

$$D_1 = G^{-1}L^{-1}K$$

NON-INTERACTING CASE

Consider Equation (33),

$$e_j^{2'} = -\sum_{i=1}^{n} g_{ji}^{-1} k_{ii} d_{ii}^2 u_i \quad :$$

from this equation it can be seen that if d_{ii}^2 contained, in its denominator, any outside poles then $e_j^{2'}$ would become unstable. In particular, d_{ii}^2 cannot contain those outside zeros of k_{ii} which were taken from g_{ji}^{-1}, as $e_j^{2'}$ would again be unstable.

Briefly the design procedure is as follows. The hold checks, condition 5, are first applied if ripple-free design is required; conditions 1, 2 and 3, and condition 6 for ripple-free design, are used to obtain the K matrix; using Equation (11), the restrictions on l_{jj} are applied and L and D_2 are found simultaneously; Equation (19) gives the elements of D_1.

These steps are now elaborated: For a ripple-free design, the first step is to apply condition 5 to the hold elements and to the elements of the inverse of the G matrix. For all j, as i = 1...n,

$h_j + 1 \geq (m_i, n_i) - f_{ji}$.

The elements of the K matrix are obtained as follows: For all j, k_{jj} must contain the maximum transport lag of $1/g_{ij}^{-1}$ as i = 1...n, from condition 2. For all j, k_{jj} must contain as its zeros the outside poles of g_{ij}^{-1}, or all the poles of g_{ij}^{-1} for a ripple-free design, as $i = 1 \dots n$. These are the same as the outside zeros, or all the zeros when ripple-free, of the determinant of G. Conditions 3 and 6 were used here. The k_{jj} obtained so far is then multiplied by

$$(a_j^o + a_j^1 z^{-1} + \dots\dots + a_j^{m_j-1} z^{-(m_j-1)}) \tag{47}$$

and condition 1 is applied: Equation (17), for all j,

$$k_{jj}(1) = 1$$

$$k'_{jj}(1) = 0$$

$$\vdots$$

$$k_{jj}^{m_j-1}(1) = 0.$$

The numerator of d_{jj}^2 is a polynomial in z^{-1} whose length is determined by n_j and the outside zeros of g_{ij}^{-1}, thus;

$$(b_j^o + b_j^1 z^{-1} + \dots + b_j^{n_j-1} z^{-(n_j-1)} + \dots + b_j^{n_j+O_j-1} z^{-(n_j+O_j-1)}) \tag{48}$$

The coefficients of this polynomial will be found later, O_j is the number of outside zeros of g_{ij}^{-1} as $i = 1 \dots n$. The denominator of d_{jj}^2 is made to be all of k_{jj}, except for the outside zeros of k_{jj}. Appendix I gives from Equation (11),

$$l_{jj} = 1 - k_{jj} d_{jj}^2 \tag{49}$$

where $k_{jj} d_{jj}^2$ has now been treated so that it is the product of the outside zeros of k_{jj} and the undetermined numerator of d_{jj}^2. If the conditions on l_{jj}, condition 4, are now applied to Equation (49), then l_{jj} and d_{jj}^2 can be found simultaneously. Condition 4 states, in the form of Equation (28), for all j,

$l_{jj}(1) = 0$

$l'_{jj}(1) = 0$

$\vdots$

$l_{jj}^{n_j - 1}(1) = 0$

l_{jj} (1^{st} outside zero of g_{ij}^{-1} as $i = 1 \ldots n$) = 0

$\vdots$

l_{jj} (O_j^{th} outside zero of g_{ij}^{-1} as $i = 1 \ldots n$) =0 .

Since both k_{jj} and l_{jj}, for all j, have now been found, d_{ij}^{1} may be directly obtained by substituting in Equation (20),

$$d_{ij}^{1} = g_{ij}^{-1} k_{jj}/l_{jj} .$$

An example of this non-interacting design method is given later to illustrate the procedure.

INTERACTING CASE

If ripple-free design is specified condition 12 must be satisfied, so that from Equation (46),

$$h_i + 1 \geq (m,n - f_i').$$

The elements of the non-diagonal K matrix are obtained in a way similar to the diagonal K matrix. The following steps are required: For all p,j, k_{pj} must contain as its zeros the maximum transport of $1/g_{ip}^{-1}$ as $i = 1 \ldots n$, from condition 8. From condition 9, additionally condition 11 for a ripple-free system, k_{pj} must contain the outside poles or all the poles for ripple-free outputs of g_{ip}^{-1} as $i = 1 \ldots n$. These poles correspond to the zeros of the determinant of G. The elements of K , so far incomplete, are then to be multiplied by the factor,

$$(a_j^o + a_j^1 z^{-1} + \ldots\ldots + a_j^{m_j - 1} z^{-(m_j - 1)}) \qquad (50)$$

and condition 7, in the form of Equations (39) and (40), is applied:
For all j,

$$k_{jj}(1) = 1$$
$$k'_{jj}(1) = 0$$
$$\vdots$$
$$k_{jj}^{m_j-1}(1) = 0$$

and for all j, as $i = 1 \ldots n$, $i \neq j$,

$$k_{ij}(1) = 0$$
$$k'_{ij}(1) = 0$$
$$\vdots$$
$$k_{ij}^{m_j-1}(1) = 0$$

So far the design has followed a method analogous to the non-interacting design, but at this stage the two design procedures must differ since K is no longer diagonal. Consider Equation (11) and in particular the matrix KD_2 which has been shown to be diagonal. The need to ensure that KD_2 is in fact diagonal is the basis of the remaining design procedure. Equation (11) is treated in Appendix I, and since the elements of KD_2 must correspond to a diagonal matrix, this treatment may be expressed as follows: For $i, j = 1 \ldots n$,

$$\sum_{p=1}^{n} k_{ip}\, d^2_{pj} = 0 \tag{51}$$

and for $j = 1 \ldots n$,

$$\sum_{p=1}^{n} k_{jp}\, d^2_{pj} = 1 - l_{jj} \tag{52}$$

From Equation (11) then, n^2 equations have been obtained. These may be broken down into n sets of similar equations with (n - 1) equations per set, those of Equation (51), and n equations obtained from Equation (52). From Equation (51), as i = 1...n, $i \neq j$,

$$-k_{i1} d_{1j}^2 = \sum_{p=2}^{n} k_{ip} d_{pj}^2 \tag{53}$$

and let j = 1...n to obtain the j^{th} set of the (n - 1) equations. Treating d_{1j}^2 for j = 1....n as a constant, then from Equation (53), there are (n - 1) equations with d_{pj}^2 as p = 2...n to be considered as unknowns. The elements of K are to be considered as constants. These (n - 1) equations can be readily solved, for example by Cramer's Rule, in terms of d_{1j}^2 for d_{pj}^2, as p = 2...n for all j. Thus for j, i = 1...n,

$$d_{ij} = \theta_{ij}(\text{k elements}, d_{1j}^2) \tag{54}$$

where θ_{ij} is found by solving the (n - 1) equations above. Equation (54) for all i, j is then substituted back into Equation (52) giving:

$$l_{jj} = 1 - \sum_{p=1}^{n} k_{jp} \theta_{pj} \tag{55}$$

or,

$$l_{jj} = 1 - d_{1j}^2 \sum_{p=1}^{n} k_{jp} \tau_{pj} \tag{56}$$

where $\tau_{pj} = \theta_{pj}/d_{1j}^2$ so that $k_{jp}\tau_{pj}$ for all j and p is only a function of all the elements of the matrix K, which of course have already been determined. The last stages of the interacting design may now be completed. For all j,

$$\beta_j = \sum_{p=1}^{n} k_{jp} \tau_{pj} \tag{57}$$

can be expanded and factorized. The denominator of β_j in its entirety is to be made the numerator of d^2_{1j} so that for all j, l_{jj} is a polynomial of finite length. The numerator of β_j, in its factored form, in general will contain inside zeros. These inside zeros are to be contained as inside poles of d^2_{1j} as $j = 1 \ldots n$. The d^2_{1j} obtained so far is then multiplied by,

$$(\alpha^o_j + \alpha^1_j z^{-1} + \ldots\ldots + \alpha_j^{n_j - 1} z^{-(n_j - 1)}) \tag{58}$$

for all j. Then,

$$l_{jj} = (\alpha^o_j + \alpha^1_j z^{-1} + \ldots\ldots + \alpha_j^{n_j - 1} z^{-(n_j - 1)})\beta'_j \tag{59}$$

where β'_j represents the remaining terms of β_j after as much as possible has been taken into d^2_{1j}. The conditions on l_{jj}, condition 10 in the form of Equation (42), are now applied to determine the values of the coefficients α^o_j, $\alpha^1_j \ldots \alpha^{n_j - 1}$ for all j. Thus l_{jj} and d^2_{1j} can be determined simultaneously for all j. The first row of the D_2 matrix has now been obtained, d^2_{1j} for all j. The remaining rows are found from Equation (54), for $j = 1 \ldots n$, $i = 2 \ldots n$,

$$d^2_{ij} = \theta_{ij} \; .$$

The elements of the D_1 matrix can be found from Equation (41),

$$d^1_{ij} = \sum_{p=1}^{n} (g^{-1}_{ip} k_{pi})/l_{pp}$$

A short example to illustrate the interacting design method will clarify this rather abstract treatment.

This concludes the design theory of the interacting system.

EXAMPLES

Two examples are given. The first example gives a complete design for the non-interacting case, for both minimal prototype and ripple free systems. The second example merely illustrates a particular point of the interacting system design, but, except for the more involved work required for the interacting system, the two systems are basically very similar.

EXAMPLE 1

Suppose the plant-hold matrix, G, for zero order holds is given below:

$$G = \left\| \begin{matrix} \dfrac{5z^{-1}}{(1 - 1.05z^{-1})} & \dfrac{3z^{-1}}{(1 - 0.1z^{-1})} \\[2ex] \dfrac{3z^{-1}}{(1 - 0.1z^{-1})} & \dfrac{2z^{-1}}{(1 - 1.05z^{-1})} \end{matrix} \right\|$$

then from Appendix II;

$$|G| = \frac{z^{-2}\,(1 + 17.45z^{-1})\,(1 - 0.555z^{-1})}{(1 - 1.05z^{-1})^2\,(1 - 0.1z^{-1})^2}$$

and thus G^{-1}

$$\left\| \begin{matrix} \dfrac{2\,(1 - 1.05z^{-1})\,(1 - 0.1z^{-1})^2}{z^{-1}\,(1 + 17.45z^{-1})\,(1 - 0.555z^{-1})} & \dfrac{-3\,(1 - 1.05z^{-1})^2\,(1 - 0.0z^{-1})}{z^{-1}\,(1 + 17.45z^{-1})\,(1 - 0.555z^{-1})} \\[2ex] \dfrac{-3\,(1 - 1.05z^{-1})^2\,(1 - 0.1z^{-1})}{z^{-1}\,(1 + 17.45z^{-1})\,(1 - 0.555z^{-1})} & \dfrac{5\,(1 - 1.05z^{-1})\,(1 - 0.1z^{-1})^2}{z^{-1}(1 + 17.45z^{-1})\,(1 - 0.555z^{-1})} \end{matrix} \right\|$$

For a minimal prototype system, assume the input and disturbance functions are of the form:

$$R = \left\| \begin{matrix} \dfrac{1}{(1 - z^{-1})} \\ \dfrac{z^{-1}}{(1 - z^{-1})^2} \end{matrix} \right\|$$

$$U = \left\| \begin{matrix} \dfrac{z^{-1}}{(1 - z^{-1})^2} \\ \dfrac{1}{(1 - z^{-1})} \end{matrix} \right\|$$

For the non-interacting design, K has been shown to be diagonal. Ripple free design is not specified so there is no need to apply the hold checks, condition 5.

The K matrix is now found from conditions 1, 2 and 3. Then k_{jj}, for all j, must contain the max. transport lag of $1/g_{ij}$ as $i = 1,.n$. Therefore k_{11} must contain z^{-1} as a factor, so also must k_{22}. For all j, k_{jj} must contain as its zeros the outside zeros of the determinant of G. Therefore k_{11} and k_{22} must contain as a factor the term $(1 + 17.45z^{-1})$. Since $m_1 = 1$, k_{11} is now multiplied by a_1^o ; and since $m_2 = 2$, k_{22} is multiplied by $(a_2^o + a_2^1 z^{-1})$. Thus,

$$k_{11} = z^{-1} (1 + 17.45z^{-1})a_1^o$$

$$k_{22} = z^{-1} (1 + 17.45z^{-1}) (a_2^o + a_2^1 z^{-1})$$

and condition 1,

$$k_{11}(1) = 1,$$

and

$$k_{22} (1) = 1$$

$$k_{22}' (1) = 0$$

is applied.

From Appendix II, these conditions give as the completed K matrix,

$$k_{11} = 0.054z^{-1}(1 + 17.45z^{-1})$$

$$k_{22} = 0.159z^{-1}(1 + 17.45z^{-1})(1 - 0.66z^{-1}) .$$

The numerator of d^2_{jj}, for all j, is a polynomial in z^{-1} whose length is determined by n_j and the outside zeros of g^{-1}_{ij}. Since $n_1 = 2$ and 0_1 is 2 (the number of outside zeros of g^{-1}_{i1} as i = 1,2). In this case the outside zero at z = 1.05 is repeated twice, and so $0_1 = 2$, then the numerator of d^2_{11} must contain 4 coefficients. Similarly, since $n_2 = 1$, and 0_2 is 2, exactly as before, then the numerator of d^2_{22} needs 3 coefficients. The denominator of d^2_{jj} is to be all of k_{jj}, except for the outside zeros of k_{jj}. Therefore according to Equation (49):

$$l_{11} = 1 - z^{-1}(1 + 17.45z^{-1})(b^o_1 + b^1_1 z^{-1} + b^2_1 z^{-2} + b^3_1 z^{-3})$$

and

$$d^2_{11} = \frac{(b^o_1 + b^1_1 z^{-1} + b^2_1 z^{-2} + b^3_1 z^{-3})}{0.054}$$

$$l_{22} = 1 - z^{-1}(1 + 17.45z^{-1})(b^o_2 + b^1_2 z^{-1} + b^2_2 z^2)$$

and

$$d^2_{22} = \frac{(b^o_2 + b^1_2 z^{-1} + b^2_2 z^{-2})}{0.159(1 - 0.66z^{-1})} .$$

If the conditions on l_{jj}, (4), for j = 1, 2 are now applied to the above equations for l_{11} and l_{22},

$$l_{11}(1) = 0$$

$$l'_{11}(1) = 0$$

$$l_{11}(1.05) = 0$$

$$l'_{11}(1.05) = 0$$

and

$$l_{22}(1) = 0$$

$$l_{22}(1.05) = 0$$

$$l'_{22}(1.05) = 0 \quad ,$$

then as shown in Appendix II the coefficients can be evaluated:

$$b_1^o = 0.54$$

$$b_1^1 = -1.12$$

$$b_1^2 = 0.674$$

$$b_1^3 = 0.22$$

$$b_2^o = 0.33$$

$$b_2^1 = -0.39$$

$$b_2^2 = 0.175 \quad .$$

Thus

$$l_{11} = (1 - z^{-1})^2 (1 - 1.05z^{-1})^2 (1 + 3.56z^{-1})$$

$$l_{22} = (1 - z^{-1})(1 - 1.05z^{-1})^2 (1 + 2.77z^{-1})$$

and

$$d_{11}^2 = 10 - 20.8z^{-1} + 12.5z^{-2} - 4.1z^{-1}$$

$$d_{22}^2 = (2.08 - 2.45z^{-1} + 1.1z^{-2}) / (1 - 0.66z^{-1}) \; ,$$

d_{ij}^1 for i, j = 1, 2 are now obtained directly from Equation (20)

$$d_{11}^1 = \frac{0.108\,(1 - 0.0z^{-1})^2}{(1 - 0.555z^{-1})(1 - z^{-1})^2 (1 - 1.05z^{-1})(1 + 3.56z^{-1})}$$

$$d_{12}^1 = \frac{-0.4777\,(1 - 0.0z^{-1})(1 - 0.66z^{-1})}{(1 - 0.555z^{-1})(1 - z^{-1})(1 + 2.777z^{-1})} \quad ,$$

$$d_{21}^{1} = \frac{-0.162\,(1 - 0.1z^{-1})}{(1 - 0.555z^{-1})\,(1 + 3.56z^{-1})\,(1 - z^{-1})^{2}}$$

$$d_{22}^{1} = \frac{0.795\,(1 - 0.0z^{-1})^{2}\,(1 - 0.66z^{-1})}{(1 - 0.555z^{-1})\,(1 - z^{-1})\,(1 - 1.05z^{-1})\,(1 + 2.77z^{-1})} \;.$$

This completes the minimal prototype controller design.

For Ripple Free Design

The hold conditions must be applied as a first step in the ripple free design: For all j, $h_j + 1 \geq (m_i, n_i) - f_{ji}$, for $i = 1 \ldots n$. With zero order holds, h_1 and h_2 are both zero, and for the particular G^{-1} matrix given in the example, f_{ji} (for all i, j) is also zero. Thus the hold conditions are not satisfied as the problem stands at the moment, as $m_2 = 2$, $n_1 = 2$. To avoid inserting more integrations in the plant or increasing the hold orders, which would make this example unnecessarily long, it is proposed that only step inputs and disturbances are used. Then,

$$R = \left\| \begin{matrix} \dfrac{1}{(1 - z^{-1})} \\ \\ \dfrac{1}{(1 - z^{-1})} \end{matrix} \right\| \qquad U = \left\| \begin{matrix} \dfrac{1}{(1 - z^{-1})} \\ \\ \dfrac{1}{(1 - z^{-1})} \end{matrix} \right\|$$

The only difference now between the minimal prototype design and the ripple free design is the inclusion of an extra restriction (condition 6) on k_{jj}. Not only must k_{jj} contain the outside poles of g_{ij}^{-1} as $i = 1 \ldots n$, it must also contain the inside zeros. The following K elements are then

$$k_{11} = z^{-1}\,(1 + 17.45z^{-1})\,(1 - 0.555z^{-1})\,a_1^{o}\,, \text{ since } m_1 = 1$$

$$k_{22} = z^{-1}\,(1 + 17.45z^{-1})\,(1 - 0.555z^{-1})\,a_2^{o}\,, \text{ since } m_2 = 2$$

and applying the conditions (1) on k_{jj} ($j = 1,2$):

$$k_{11} = k_{22} = 0.1222z^{-1}\,(1 + 17.45z^{-1})\,(1 - 0.555z^{-1})\ ,$$

l_{11} and l_{22} are evidently the same as l_{22} in the above minimal prototype case:

$$l_{11} = l_{22} = (1 - z^{-1})\,(1 - 1.05z^{-1})^2\,(1 + 2.77z^{-1})\ ,$$

but d_{11}^2 and d_{22}^2 are, of course, different from the minimal prototype.

$$d_{11}^2 = \frac{(2.7 - 3.2z^{-1} + 1.44z^2)}{(1 - 0.555z^{-1})}$$

$$d_{22}^2 = \frac{(4.4 - 9.2z^{-1} + 5.5z^{-2} - 1.8z^{-3})}{(1 - 0.555z^{-1})}$$

The elements of D_1 are found as before:

$$d_{11}^1 = \frac{(1 - 0.1z^{-1})^2 \;\; 0.244}{(1 - z^{-1})\,(1 - 1.05z^{-1})\,(1 + 2.77z^{-1})}$$

$$d_{21}^1 = -\frac{0.366\,(1 - 0.0z^{-1})}{(1 - z^{-1})\,(1 + 2.77z^{-1})}$$

$$d_{22}^1 = \frac{0.61\,(1 - 0.1z^{-1})^2}{(1 - z^{-1})\,(1 - 1.05z^{-1})\,(1 + 2.77z^{-1})}\ .$$

The outputs, C and C' can be directly calculated from Equations (1) and (3).

For the minimal prototype system:

$c_1 = 0.054z^{-1} + z^{-2} + z^{-3} + \ldots$ step

$c_2 = 0.159z^{-2} + 3z^{-3} + 4z^{-4} + \ldots$ ramp

$c_1' = z^{-1} + 1.46z^{-2} - 6.4z^{-3} + 3.9z^{-4}\ldots$ ramp disturbance

$c_2' = 1 + 0.67z^{-1} - 4.7z^{-1} + 3.04z^{-3}\ldots$ step disturbance.

For the ripple free system:

$$c_1 = 0.122z^{-1} + 2.180z^{-2} + z^{-3} + \ldots \qquad \text{step}$$

$$c_2 = 0.122z^{-1} + 2.182z^{-2} + z^{-3} + \ldots \qquad \text{step}$$

$$c_1' = 1 + 0.67z^{-1} - 4.7z^{-2} + 3.04z^{-3} \ldots \qquad \text{step}$$

$$c_2' = 1 + 0.67z^{-1} - 4.7z^{-2} + 3.04z^{-3} \ldots \qquad \text{step} .$$

To aid the interpretation of these outputs, they are plotted graphically; Figures 2, 3, 4 and 5. It can be seen from Fig. 3 , the minimal prototype disturbances, that the output has an initially large fluctuation. If this is considered undesirable, then extra constants may be included in the appropriate element of L, and then adjusted to minimize the magnitude of the fluctuations at the output. These extra constants will of course entail a longer settling time, but this may not be too important.

EXAMPLE 2

To help clarify the interacting design method, an abreviated example is given. Assume for a 2 by 2 system, the K matrix has already been designed, then, with D_2 and L to be determined, the diagonal matrix,

$$KD_2 = \begin{Vmatrix} (k_{11}d_{11}^2 + k_{12}d_{21}^2) & (k_{11}d_{12}^2 + k_{12}d_{22}^2) \\ (k_{22}d_{11}^2 + k_{22}d_{21}^2) & (k_{21}d_{12}^2 + k_{22}d_{22}^2) \end{Vmatrix}$$

so that, from Equations (51) and (52),

$$l_{11} = 1 - (k_{11}d_{11}^2 + k_{12}d_{21}^2) \tag{60}$$

$$0 = k_{11}d_{12}^2 + k_{12}d_{22}^2 \tag{61}$$

$$0 = k_{21}d_{11}^2 + k_{22}d_{21}^2 \tag{62}$$

$$l_{22} = 1 - (k_{21}d_{12}^2 + k_{22}d_{22}^2) . \tag{63}$$

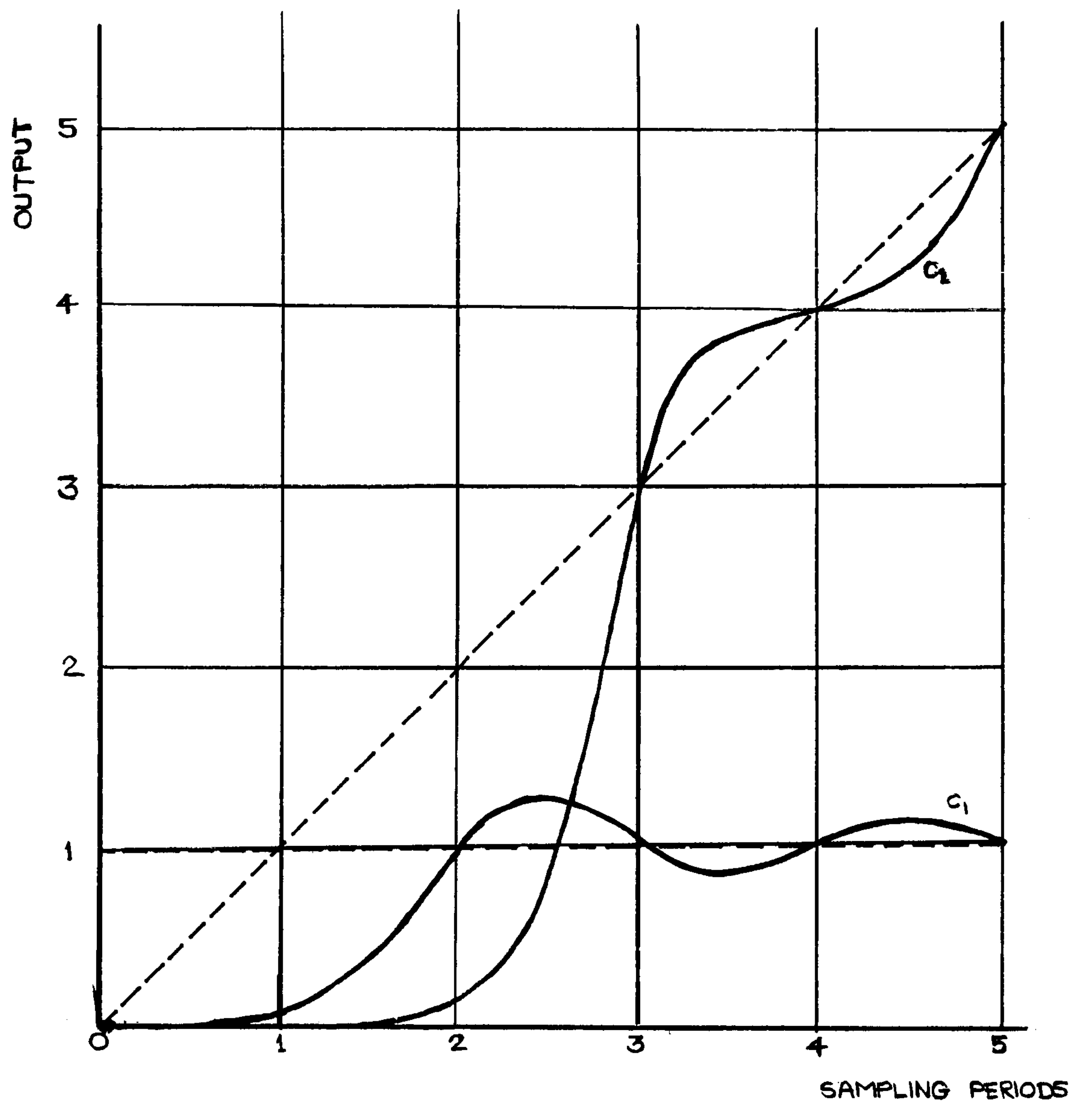

Figure 2. Outputs c_1 and c_2 for a minimal prototype system

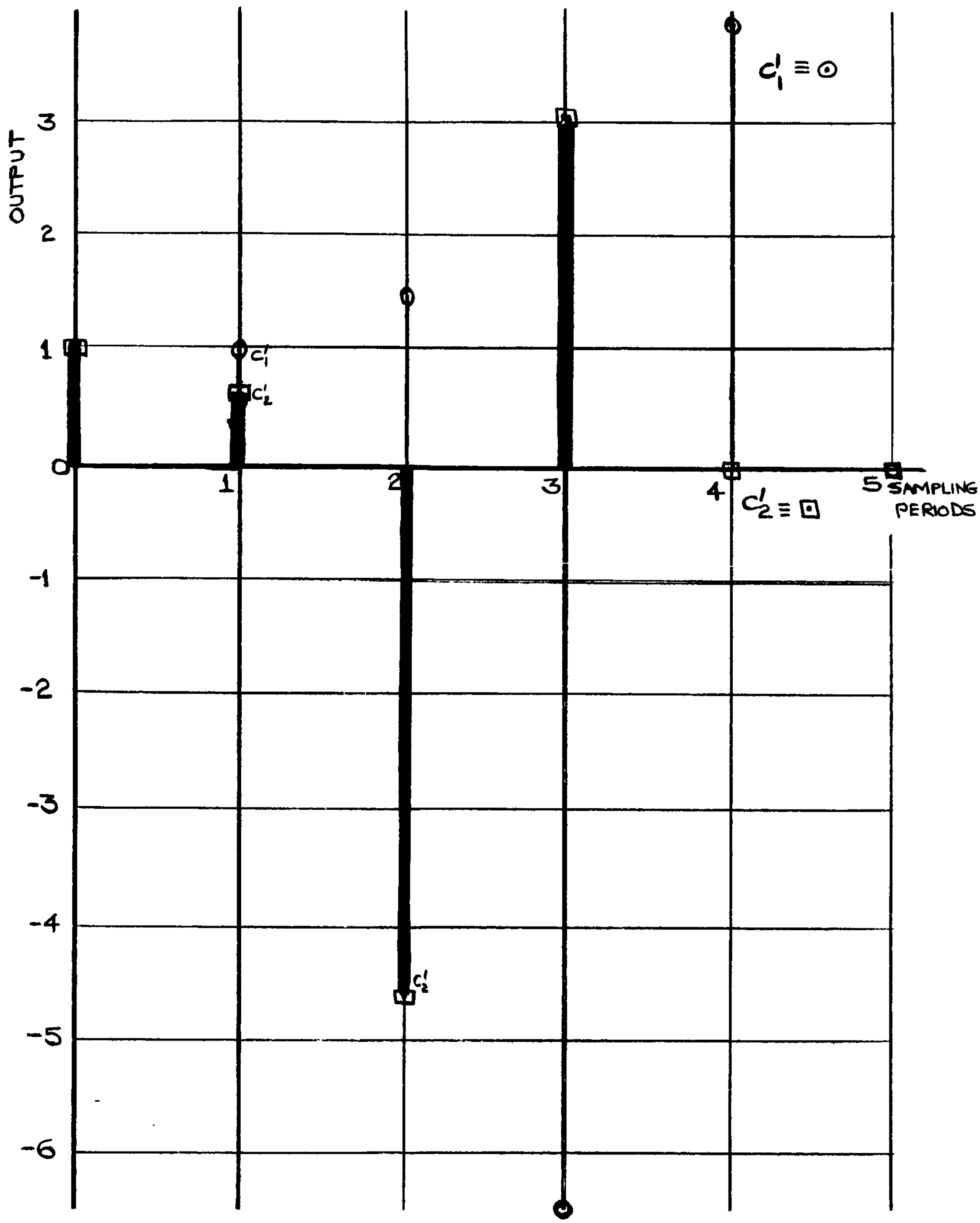

Figure 3. Outputs c_1' and c_2' for a minimal prototype system

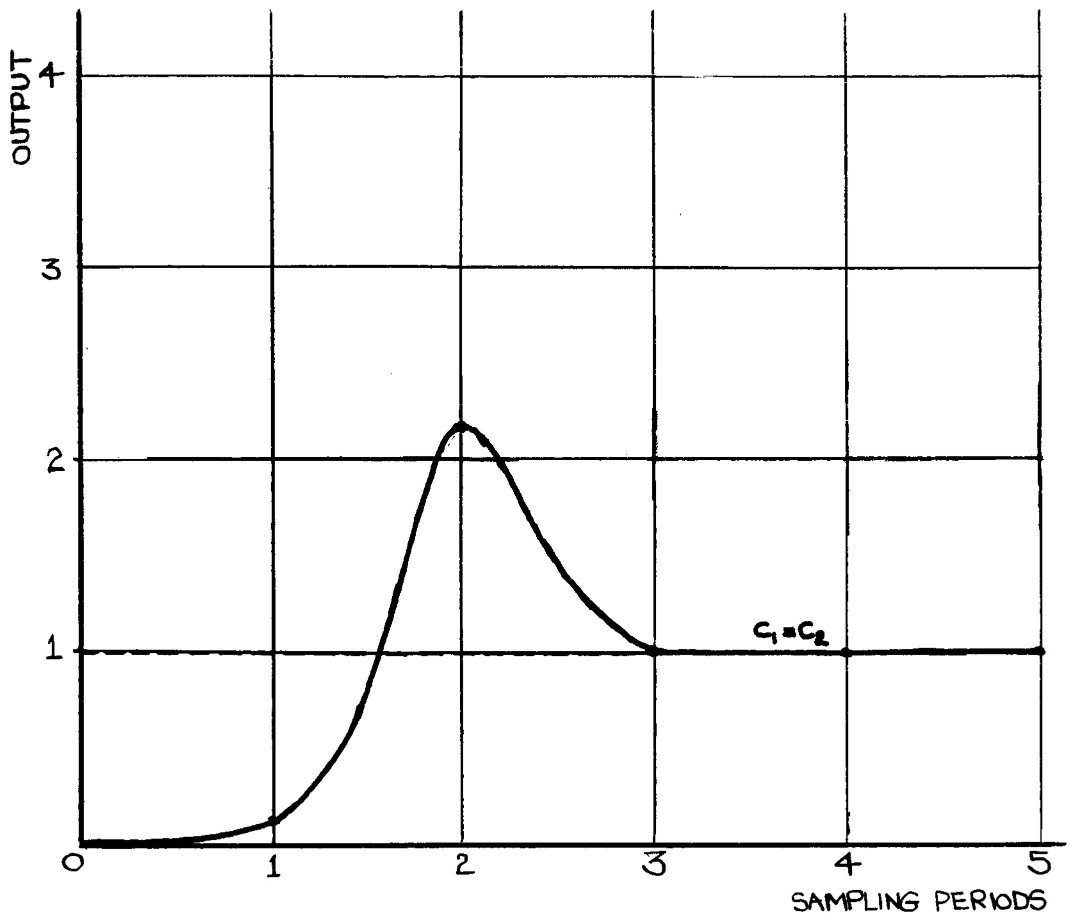

Figure 4. Outputs c_1 and c_2 for a ripple-free system

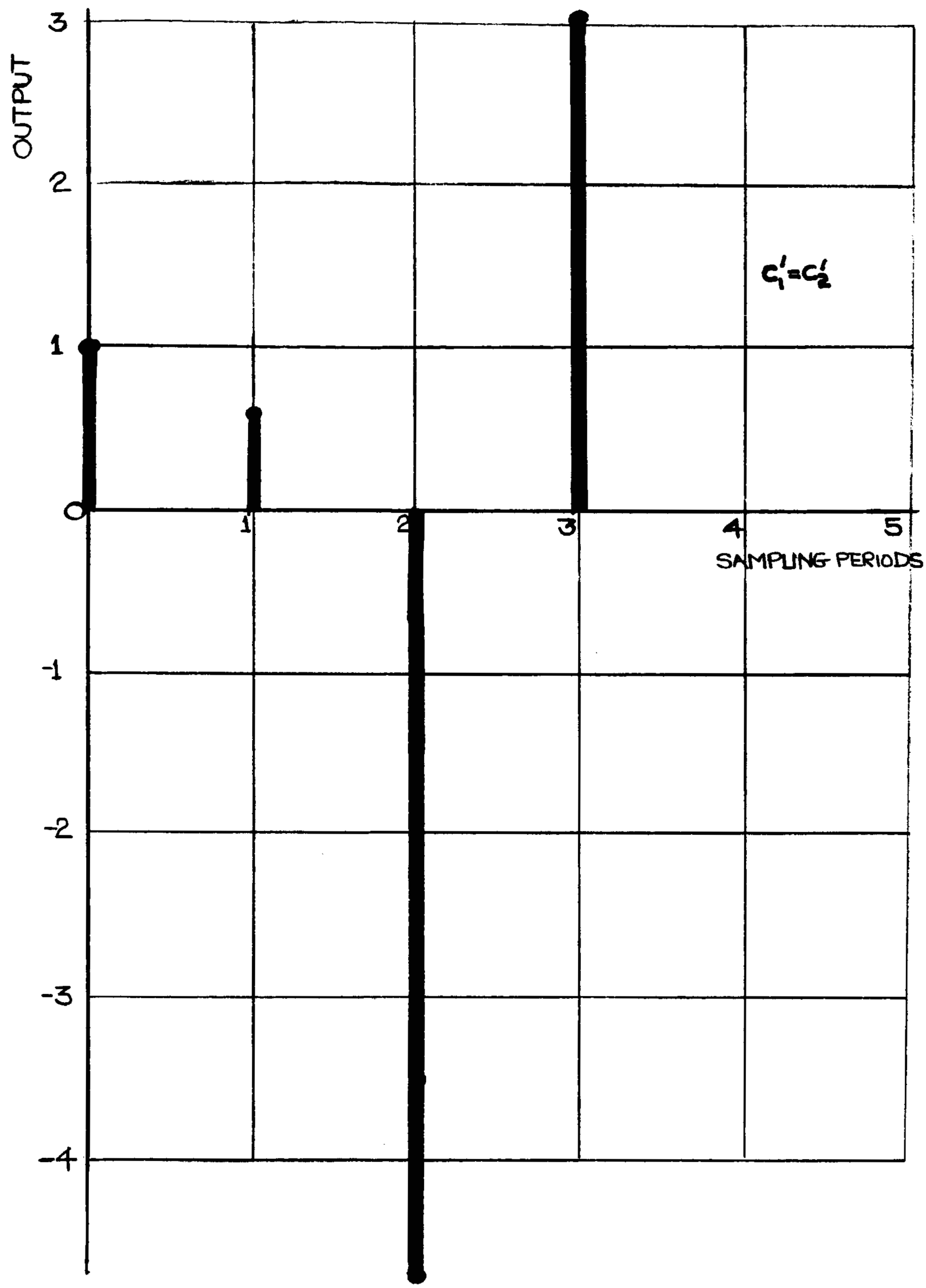

Figure 5. Outputs c_1' and c_2' for a ripple-free system

Then Equations (61) and (62) give d^2_{21} and d^2_{22}, as explained in the interacting design method. Then

$$d^2_{21} = - \frac{k_{21}d^2_{11}}{k_{22}} ,$$

$$d^2_{22} = - \frac{k_{11}d^2_{12}}{k_{12}}$$

and substituting these back into Equations (60) and (63), then the following relations are obtained,

$$l_{11} = 1 - (k_{11}d^2_{11} + k_{12} (- \frac{k_{21}d^2_{11}}{k_{22}}))$$

$$l_{22} = 1 - (k_{21}d^2_{12} + k_{22} (- \frac{k_{11}d^2_{12}}{k_{12}}))$$

which lead directly to

$$l_{11} = 1 - \frac{d^2_{11}}{k_{22}} (k_{11}k_{22} - k_{12}k_{21})$$

$$l_{22} = 1 - \frac{d^2_{12}}{k_{12}} (k_{21}k_{12} - k_{22}k_{11}).$$

Continuing the design procedure, then k_{22} and k_{12} are made the numerator of d^2_{11} and d^2_{12} respectively. Next $(k_{11}k_{22} - k_{12}k_{21})$ and $(k_{21}k_{12} - k_{22}k_{11})$ are expanded and factorized and as much as is physically realizable and stable are to be contained in the denominator of their respective D_2 elements. The remainder of the numerator of d^2_{11} and d^2_{12} is composed of the polynomial in z^{-1}, whose coefficients are to be determined by applying the l_{jj} restrictions Equations (59) and (42). Finally, having found the first row of the D_2 matrix, the remaining elements may be found from:

$$d_{22}^2 = -\frac{k_{11}d_{12}}{k_{12}}$$

$$d_{21}^2 = -\frac{k_{21}d_{11}}{k_{22}} .$$

The remainder of the interacting system design needs no further clarification.

CONCLUSIONS

This paper has presented the theory and procedure used to design digital controllers for multivariable sampled-data control systems. The theory of the design method is based extensively on a paper by Hung[5]. Simple examples to demonstrate the design procedure practically have been given.

The essence of the method is to incorporate fundamentally in the K and L matrices, the basic design criteria of multivariable sampled-data control systems: Interaction and non-interaction, minimal prototype and ripple free outputs.

By using the overall transmission matrices K and L as a basis, the procedure developed ensures awareness of the actual multivariable concept. This may be compared favorably with Sobral's method[7] of designing controllers for multivariable systems by breaking down the system into individual single variable systems.

A further advantage demonstrated is that unstable poles of the plant do not place restrictions on the K matrix; this gives a faster response than that available with the controller configuration of Sobral (with both controllers in the forward loop).

Because the design procedure is based only upon stability restrictions on K and L and the necessary physical realizability of the controllers, the design method becomes almost mechanical and is very simple to use.

The design system readily allows the treatment of systems with more outputs than there are inputs. The dimension of the matrices used in this case corresponds to the number of outputs, and the missing input or disturbance elements are merely designated by zero elements in their matrix. Similarly if there are more inputs than outputs, then the fictitious outputs may be established.

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APPENDICES

APPENDIX I: ELEMENT EQUATIONS FROM MATRIX EQUATIONS

$D_1 = G^{-1}L^{-1}K$ where K is a diagonal matrix.

$$\left\| \begin{matrix} d^1_{11} & d^1_{12} & d^1_{13} \\ d^1_{21} & d^1_{22} & d^1_{23} \\ d^1_{31} & d^1_{32} & d^1_{33} \end{matrix} \right\| = \left\| \begin{matrix} g^{-1}_{11} & g^{-1}_{12} & g^{-1}_{13} \\ g^{-1}_{21} & g^{-1}_{22} & g^{-1}_{23} \\ g^{-1}_{31} & g^{-1}_{32} & g^{-1}_{33} \end{matrix} \right\| \left\| \begin{matrix} 1/l_{11} & 0 & 0 \\ 0 & 1/l_{22} & 0 \\ 0 & 0 & 1/l_{33} \end{matrix} \right\| \left\| \begin{matrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{matrix} \right\|$$

$$= \left\| \begin{matrix} g^{-1}_{11} & g^{-1}_{12} & g^{-1}_{13} \\ g^{-1}_{21} & g^{-1}_{22} & g^{-1}_{23} \\ g^{-1}_{31} & g^{-1}_{32} & g^{-1}_{33} \end{matrix} \right\| \left\| \begin{matrix} k_{11}/l_{11} & 0 & 0 \\ 0 & k_{22}/l_{22} & 0 \\ 0 & 0 & k_{33}/l_{33} \end{matrix} \right\|$$

$$= \left\| \begin{matrix} g^{-1}_{11} k_{11}/l_{11} & g^{-1}_{12} k_{22}/l_{22} & g^{-1}_{13} k_{33}/l_{33} \\ g^{-1}_{21} k_{11}/l_{11} & g^{-1}_{22} k_{22}/l_{22} & g^{-1}_{23} k_{33}/l_{33} \\ g^{-1}_{31} k_{11}/l_{11} & g^{-1}_{32} k_{22}/l_{22} & g^{-1}_{33} k_{33}/l_{33} \end{matrix} \right\|$$

in general,

$$d^1_{ij} = g^{-1}_{ij} k_{jj}/l_{jj}$$

$E_2 = D_1 LR$

$$\begin{Vmatrix} e_1^2 \\ e_2^2 \\ e_3^2 \end{Vmatrix} = \begin{Vmatrix} d_{11}^1 & d_{12}^1 & d_{13}^1 \\ d_{21}^1 & d_{22}^1 & d_{23}^1 \\ d_{31}^1 & d_{32}^1 & d_{33}^1 \end{Vmatrix} \begin{Vmatrix} l_{11} & 0 & 0 \\ 0 & l_{22} & 0 \\ 0 & 0 & l_{33} \end{Vmatrix} \begin{Vmatrix} r_1 \\ r_2 \\ r_3 \end{Vmatrix}$$

$$= \begin{Vmatrix} d_{11}^1 & d_{12}^1 & d_{13}^1 \\ d_{21}^1 & d_{22}^1 & d_{23}^1 \\ d_{31}^1 & d_{32}^1 & d_{33}^1 \end{Vmatrix} \begin{Vmatrix} l_{11} r_1 \\ l_{22} r_2 \\ l_{33} r_3 \end{Vmatrix}$$

$$= \begin{Vmatrix} (d_{11}^1 \, l_{11} \, r_1 + d_{12}^1 \, l_{22} \, r_2 + d_{13}^1 \, l_{33} \, r_3) \\ (d_{21}^1 \, l_{11} \, r_1 + d_{22}^1 \, l_{22} \, r_2 + d_{23}^1 \, l_{33} \, r_3) \\ (d_{31}^1 \, l_{11} \, r_1 + d_{32}^1 \, l_{22} \, r_2 + d_{33}^1 \, l_{33} \, r_3) \end{Vmatrix}$$

in general,

$$e_j^2 = \sum_{p=1}^{n} d_{jp}^1 \, l_{pp} \, r_p$$

$E_2 = G^{-1}KR$ where K is diagonal.

$$\begin{Vmatrix} e_1^2 \\ e_2^2 \\ e_3^2 \end{Vmatrix} = \begin{Vmatrix} g_{11}^{-1} & g_{12}^{-1} & g_{13}^{-1} \\ g_{21}^{-1} & g_{22}^{-1} & g_{23}^{-1} \\ g_{31}^{-1} & g_{32}^{-1} & g_{33}^{-1} \end{Vmatrix} \begin{Vmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{Vmatrix} \begin{Vmatrix} r_1 \\ r_2 \\ r_3 \end{Vmatrix}$$

$$= \begin{Vmatrix} g_{11}^{-1} & g_{12}^{-1} & g_{13}^{-1} \\ g_{21}^{-1} & g_{22}^{-1} & g_{23}^{-1} \\ g_{31}^{-1} & g_{32}^{-1} & g_{33}^{-1} \end{Vmatrix} \begin{Vmatrix} k_{11}r_1 \\ k_{22}r_2 \\ k_{33}r_3 \end{Vmatrix}$$

$$= \begin{Vmatrix} (g_{11}^{-1}k_{11}r_1 + g_{12}^{-1}k_{22}r_2 + g_{13}^{-1}k_{33}r_3) \\ (g_{21}^{-1}k_{11}r_1 + g_{22}^{-1}k_{22}r_2 + g_{23}^{-1}k_{33}r_3) \\ (g_{31}^{-1}k_{11}r_1 + g_{32}^{-1}k_{22}r_2 + g_{33}^{-1}k_{33}r_3) \end{Vmatrix}$$

in general:

$$e_j^2 = \sum_{i=1}^{n} g_{ji}^{-1} k_{ii}\, r_i$$

$E_2' = -G^{-1}KD_2U$ where K and D_2 are diagonal.

$$\begin{Vmatrix} e_1^{2'} \\ e_2^{2'} \\ e_2^{2'} \end{Vmatrix} = - \begin{Vmatrix} g_{11}^{-1} & g_{12}^{-1} & g_{13}^{-1} \\ g_{21}^{-1} & g_{22}^{-1} & g_{23}^{-1} \\ g_{31}^{-1} & g_{32}^{-1} & g_{33}^{-1} \end{Vmatrix} \begin{Vmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{Vmatrix} \begin{Vmatrix} d_{11}^2 & 0 & 0 \\ 0 & d_{22}^2 & 0 \\ 0 & 0 & d_{33}^2 \end{Vmatrix} \begin{Vmatrix} u_1 \\ u_2 \\ u_3 \end{Vmatrix}$$

$$= - \begin{Vmatrix} g_{11}^{-1} & g_{12}^{-1} & g_{13}^{-1} \\ g_{21}^{-1} & g_{22}^{-1} & g_{23}^{-1} \\ g_{31}^{-1} & g_{32}^{-1} & g_{33}^{-1} \end{Vmatrix} \begin{Vmatrix} k_{11}\, d_{11}^2\, u_1 \\ k_{22}\, d_{22}^2\, u_2 \\ k_{33}\, d_{33}^2\, u_3 \end{Vmatrix}$$

$$= - \begin{Vmatrix} (g_{11}^{-1} k_{11} d_{11}^2 u_1 + g_{12}^{-1} k_{22} d_{22}^2 u_2 + g_{13}^{-1} k_{33} d_{33}^2 u_3) \\ (g_{21}^{-1} k_{11} d_{11}^2 u_1 + g_{22}^{-1} k_{22} d_{22}^2 u_2 + g_{23}^{-1} k_{33} d_{33}^2 u_3) \\ (g_{31}^{-1} k_{11} d_{11}^2 u_1 + g_{32}^{-1} k_{22} d_{22}^2 u_2 + g_{33}^{-1} k_{33} d_{33}^2 u_3) \end{Vmatrix}$$

in general,

$$e_j^{2'} = - \sum_{i=1}^{n} g_{ji}^{-1} k_{ii}\, d_{ii}^2\, u_i$$

$E = \{I - K\} R$ where K is non-diagonal.

$$\begin{Vmatrix} e_1 \\ e_2 \\ e_3 \end{Vmatrix} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} - \begin{Vmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{Vmatrix} \times \begin{Vmatrix} r_1 \\ r_2 \\ r_3 \end{Vmatrix}$$

$$= \begin{Vmatrix} (1 - k_{11}) & -k_{12} & -k_{13} \\ -k_{21} & (1 - k_{22}) & -k_{23} \\ -k_{31} & -k_{32} & (1 - k_{33}) \end{Vmatrix} \begin{Vmatrix} r_1 \\ r_2 \\ r_3 \end{Vmatrix}$$

$$= \begin{Vmatrix} (1 - k_{11})r_1 - k_{12}r_2 - k_{13}r_3 \\ -k_{21} + (1 - k_{22})r_2 - k_{23}r_3 \\ -k_{31}r_1 - k_{32}r_2 + (1 - k_{33})r_3 \end{Vmatrix}$$

in general:

$$e_j = (1 - k_{jj})r_j - \sum_{\substack{i = 1 \\ i \neq j}}^{n} k_{ji}\, r_i .$$

If K is diagonal then k_{ij} for $i \neq j$ is zero, therefore for K diagonal,

$$e_j = (1 - k_{jj})r_j$$

$D_1 = G^{-1} L^{-1} K$ where K is non-diagonal.

$$\begin{Vmatrix} d^1_{11} & d^1_{12} & d^1_{13} \\ d^1_{21} & d^1_{22} & d^1_{23} \\ d^1_{31} & d^1_{32} & d^1_{33} \end{Vmatrix} = \begin{Vmatrix} g^{-1}_{11} & g^{-1}_{12} & g^{-1}_{13} \\ g^{-1}_{21} & g^{-1}_{22} & g^{-1}_{23} \\ g^{-1}_{31} & g^{-1}_{32} & g^{-1}_{33} \end{Vmatrix} \begin{Vmatrix} 1/l_{11} & 0 & 0 \\ 0 & 1/l_{22} & 0 \\ 0 & 0 & 1/l_{33} \end{Vmatrix} \begin{Vmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{Vmatrix}$$

$$= \begin{Vmatrix} g^{-1}_{11} & g^{-1}_{12} & g^{-1}_{13} \\ g^{-1}_{21} & g^{-1}_{22} & g^{-1}_{23} \\ g^{-1}_{31} & g^{-1}_{32} & g^{-1}_{33} \end{Vmatrix} \begin{Vmatrix} k_{11}/l_{11} & k_{12}/l_{11} & k_{13}/l_{11} \\ k_{21}/l_{22} & k_{22}/l_{22} & k_{23}/l_{22} \\ k_{31}/l_{33} & k_{32}/l_{33} & k_{33}/l_{33} \end{Vmatrix}$$

$$= \begin{Vmatrix} (g^{-1}_{11} k_{11}/l_{11} + g^{-1}_{12} k_{21}/l_{22} + g^{-1}_{13} k_{31}/l_{33}) & (d^1_{12}) & (d^1_{13}) \\ (g^{-1}_{21} k_{11}/l_{11} + g^{-1}_{22} k_{21}/l_{22} + g^{-1}_{23} k_{31}/l_{33}) & (d^1_{22}) & (d^1_{23}) \\ (d^1_{31}) & (g^{-1}_{31} k_{12}/l_{11} + g^{-1}_{32} k_{22}/l_{22} + g^{-1}_{33} k_{32}/l_{33}) & (d^1_{33}) \end{Vmatrix}$$

in general:

$$d^1_{ij} = \sum_{p=1}^{n} g^{-1}_{ip} k_{pj}/l_{pp}$$

$E_2 = G^{-1}KR$ where K is non-diagonal.

$$\begin{Vmatrix} e_1^2 \\ e_2^2 \\ e_3^2 \end{Vmatrix} = \begin{Vmatrix} g_{11}^{-1} & g_{12}^{-1} & g_{13}^{-1} \\ g_{21}^{-1} & g_{22}^{-1} & g_{23}^{-1} \\ g_{31}^{-1} & g_{32}^{-1} & g_{33}^{-1} \end{Vmatrix} \begin{Vmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{Vmatrix} \begin{Vmatrix} r_1 \\ r_2 \\ r_3 \end{Vmatrix}$$

$$= \begin{Vmatrix} g_{11}^{-1} & g_{12}^{-1} & g_{13}^{-1} \\ g_{21}^{-1} & g_{22}^{-1} & g_{23}^{-1} \\ g_{31}^{-1} & g_{32}^{-1} & g_{33}^{-1} \end{Vmatrix} \begin{Vmatrix} k_{11}r_1 + k_{12}r_2 + k_{13}r_3 \\ k_{21}r_1 + k_{22}r_2 + k_{23}r_3 \\ k_{31}r_1 + k_{32}r_2 + k_{33}r_3 \end{Vmatrix}$$

$$= \begin{Vmatrix} g_{11}^{-1}(k_{11}r_1+k_{12}r_2+k_{13}r_3) + g_{12}^{-1}(k_{21}r_1+k_{22}r_2+k_{23}r_3) + g_{13}^{-1}(k_{31}r_1+k_{32}r_2+k_{33}r_3) \\ g_{21}^{-1}(k_{11}r_1+k_{12}r_2+k_{13}r_3) + g_{22}^{-1}(k_{21}r_1+k_{22}r_2+k_{23}r_3) + g_{23}^{-1}(k_{31}r_1+k_{32}r_2+k_{33}r_3) \\ g_{31}^{-1}(k_{11}r_1+k_{12}r_2+k_{13}r_3) + g_{32}^{-1}(k_{21}r_1+k_{22}r_2+k_{23}r_3) + g_{33}^{-1}(k_{31}r_1+k_{32}r_2+k_{33}r_3) \end{Vmatrix}$$

in general:

$$e_j^2 = \sum_{q=1}^{n} g_{jq}^{-1} \sum_{p=1}^{n} k_{qp}\, r_p$$

$L = I - KD_2$ where both K and D_2 are diagonal.

$$\begin{Vmatrix} l_{11} & 0 & 0 \\ 0 & l_{22} & 0 \\ 0 & 0 & l_{33} \end{Vmatrix} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} - \begin{Vmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & 0 \\ 0 & 0 & k_{33} \end{Vmatrix} \begin{Vmatrix} d_{11}^2 & 0 & 0 \\ 0 & d_{22}^2 & 0 \\ 0 & 0 & d_{33}^2 \end{Vmatrix}$$

$$= \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} - \begin{Vmatrix} k_{11}d_{11}^2 & 0 & 0 \\ 0 & k_{22}d_{22}^2 & 0 \\ 0 & 0 & k_{33}d_{33}^2 \end{Vmatrix}$$

$$= \begin{Vmatrix} 1 - k_{11}d_{11}^2 & 0 & 0 \\ 0 & 1 - k_{22}d_{22}^2 & 0 \\ 0 & 0 & 1 - k_{33}d_{33}^2 \end{Vmatrix}$$

in general:

$$l_{jj} = 1 - k_{jj}\, d_{jj}^2$$

$L = I - KD_2$ where both K and D_2 are non-diagonal.

$$KD_2 = \left\| \begin{matrix} (k_{11}d_{11}^2+k_{12}d_{21}^2+k_{13}d_{31}^2) & (k_{11}d_{12}^2+k_{12}d_{22}^2+k_{13}d_{32}^2) & (k_{11}d_{13}^2+k_{12}d_{23}^2+k_{13}d_{33}^2) \\ (k_{21}d_{11}^2+k_{22}d_{21}^2+k_{23}d_{31}^2) & (k_{21}d_{12}^2+k_{22}d_{22}^2+k_{23}d_{32}^2) & (k_{21}d_{13}^2+k_{22}d_{23}^2+k_{23}d_{33}^2) \\ (k_{31}d_{11}^2+k_{32}d_{21}^2+k_{33}d_{31}^2) & (k_{31}d_{12}^2+k_{32}d_{22}^2+k_{33}d_{32}^2) & (k_{31}d_{13}^2+k_{32}d_{23}^2+k_{33}d_{33}^2) \end{matrix} \right\|$$

and the individual elements of $I - KD_2$ are then:

element, 11 $= 1 - (k_{11}d_{11}^2 + k_{12}d_{21}^2 + k_{13}d_{31}^2)$

12 $= -(k_{11}d_{12}^2 + k_{12}d_{22}^2 + k_{13}d_{32}^2) = 0$

13 $= -(k_{11}d_{13}^2 + k_{12}d_{23}^2 + k_{13}d_{33}^2) = 0$

21 $= -(k_{21}d_{11}^2 + k_{22}d_{21}^2 + k_{23}d_{31}^2) = 0$

22 $= 1 - (k_{21}d_{12}^2 + k_{22}d_{22}^2 + k_{23}d_{32}^2)$

23 $= -(k_{21}d_{13}^2 + k_{22}d_{23}^2 + k_{23}d_{33}^2) = 0$

31 $= -(k_{31}d_{11}^2 + k_{32}d_{21}^2 + k_{33}d_{31}^2) = 0$

32 $= -(k_{31}d_{12}^2 + k_{32}d_{22}^2 + k_{33}d_{32}^2) = 0$

33 $= 1 - (k_{31}d_{13}^2 + k_{32}d_{23}^2 + k_{33}d_{33}^2)$

in general the diagonal elements of $I - KD_2$ are given by:

$$l_{jj} = (\text{element})_{jj} = 1 - \sum_{p=1}^{n} k_{jp}\, d_{pj}^2$$

and for the off-diagonal elements:

$$0 = \underset{i \neq j}{(\text{element})_{ij}} = - \sum_{p=1}^{n} k_{ip} \, d^2_{pj}$$

APPENDIX II. CALCULATIONS FOR EXAMPLE I

$$|G| = \frac{10z^{-2}}{(1 - 1.05z^{-1})^2} - \frac{9z^{-2}}{(1 - 0.1z^{-1})^2}$$

$$= \frac{10z^{-2}(1 - 0.1z^{-1})^2 - 9z^{-2}(1 - 1.05z^{-1})^2}{(1 - 1.05z^{-1})^2(1 - 0.1z^{-1})^2}$$

$$= \frac{10z^{-2}(1 - 0.2z^{-1} + 0.01z^{-2}) - 9z^{-2}(1 - 2.1z^{-1} + 1.103z^{-2})}{(1 - 1.05z^{-1})^2(1 - 0.1z^{-1})^2}$$

$$= \frac{z^{-2}(10 - 2z^{-1} + 0.1z^{-2} - 9 + 18.9z^{-1} - 9.9225z^{-2})}{(1 - 1.05z^{-1})^2(1 - 0.1z^{-1})^2}$$

$$= \frac{z^{-2}(1 + 16.9z^{-1} - 9.82z^{-2})}{(1 - 1.05z^{-1})^2(1 - 0.1z^{-1})^2}$$

$$= \frac{z^{-2}(1 + 17.45z^{-1})(1 - 0.555z^{-1})}{(1 - 1.05z^{-1})^2(1 - 0.1z^{-1})^2}$$

The elements of K are now treated; from the example,

$$k_{11} = z^{-1}(1 + 17.45z^{-1})a_1^o$$

and $k_{11}(1) = 1$, so that,

$$1 = (1 + 17.45z^{-1})a_1^o,$$

therefore,

$$a_1^o = 1/18.45 = 0.054 .$$

Also,

$$k_{22} = z^{-1}(1 + 17.45z^{-1})(a_2^o + a_2^1 z^{-1})$$

and,

$$k_{22}(1) = 1$$

$$k'_{22}(1) = 0$$

Expanding k_{22}:

$$k_{22} = z^{-1}(a_2^o + a_2^1 z^{-1} + 17.45a_2^o z^{-1} + 17.45a_2^1 z^{-2})$$
$$= a_2^o z^{-1} + z^{-2}(a_2^1 + 17.45a_2^o) + 17.45a_2^1 z^{-3}.$$
$$k'_{22} = a_2^o + 2z^{-1}(a_2^1 + 17.45a_2^o) + 3(17.45a_2^1 z^{-2})$$
$$k'_{22}(1) = a_2^o + 2a_2^1 + 34.9a_2^o + 52.35a_2^1$$
$$= 35.9a_2^o + 54.35a_2^1$$

Now applying the conditions,

$$1 = 18.45(a_2^o + a_2^1)$$
$$0 = 35.9a_2^o + 54.35a_2^1$$

rearranging,

$$0 = 35.9a_2^o + 54.35a_2^1$$
$$1.95 = 35.9a_2^o + 35.9a_2^1$$

and subtracting,

$$-1.95 = 18.45a_2^1$$

therefore,

$$a_2^1 = -1.05$$
$$a_2^o = 0.159.$$

Therefore,

$$k_{22} = 0.159z^{-1}(1 + 17.45z^{-1})(1 - 0.66z^{-1}).$$

Now l_{11} and l_{22} are calculated: Since the factors which must be contained in these elements are known, the determination of the "b" coefficients may be much simplified by equating coefficients of like powers of z^{-1}.

Given,

$$l_{11} = 1 - z^{-1}(1 + 17.45z^{-1})(b_1^o + b_1^1 z^{-1} + b_1^2 z^{-2} + b_1^3 z^{-3}) \qquad \text{(A1)}$$

and the conditions;

$$l_{11}(1) = 0$$
$$l'_{11}(1) = 0$$
$$l'_{11}(1.05) = 0$$
$$l_{11}(1.05) = 0,$$

it is required to find the coefficients b_1^o, b_1^1, b_1^2 and b_1^3. This means that l_{11} must be,

$$l_{11} = (1 - z^{-1})^2(1 - 1.05z^{-1})^2(1 - \gamma z^{-1}) \qquad \text{(A2)}$$

where γ is to be determined.

Expanding Equations (A1) and (A2);

$$l_{11} = 1 - b_1^o z^{-1} - z^{-2}(b_1^1+17.45b_1^o)-z^{-3}(b_1^2+17.45b_1^1)-z^{-4}(b_1^3+17.45b_1^2)$$
$$-z^{-5}17.45b_1^3 ,$$

$$l_{11} = 1 - z^{-1}(4.1-\gamma) -z^{-2}(4.1\gamma-6.3) -z^{-3}(4.3-6.3\gamma) -z^{-4}(4.3\gamma-1.1)+1.1\gamma z^{-5}.$$

Comparing coefficients of like powers of z^{-1}:

$$4.1 - \gamma = b_1^o$$
$$4.1\gamma - 6.3 = b_1^1 + 17.45b_1^o$$
$$4.3 - 6.3\gamma = b_1^2 + 17.45b_1^1$$
$$4.3\gamma - 1.1 = b_1^3 + 17.45b_1^2$$
$$1.1\gamma \quad = -17,45b_1^3.$$

These are very easily solved, and give,

$$\gamma = 3.56.$$

Substituting this value for γ the coefficients can be found;

$$b_1^o = 0.54$$
$$b_1^1 = -1.12$$
$$b_1^2 = 0.674$$
$$b_1^3 = -0.22,$$

and l_{11} is;

$$l_{11} = (1 - z^{-1})^2(1 - 1.05z^{-1})^2(1 + 3.56z^{-1})$$

Again given,

$$l_{22} = 1 -z^{-1}(1 + 17.45z^{-1})(b_2^o + b_2^o z^{-1} + b_2^2 z^{-2}) \qquad \text{(A3)}$$

and the conditions;

$$l_{22}(1) = 0$$
$$l_{22}(1.05) = 0$$
$$l_{22}'(1.05) = 0,$$

a similar procedure may be repeated for l_{22}.

l_{22} is also given by,

$$l_{22} = (1 - z^{-1})(1 - 1.05z^{-1})^2(1 + \mu z^{-1}) \tag{A4}$$

where μ is to be determined.

Expanding Equations (A3) and (A4),

$$l_{11} = 1 - z^{-1}b_2^o - z^{-2}(b_2^1 + 17.45b_2^o) - z^{-3}(b_2^2 + 17.45b_2^1) - z^{-4}17.45b_2^2$$

$$l_{11} = 1 - z^{-1}(3.1 - \mu) - z^{-2}(3.1\mu - 3.2) - z^{-3}(1.1 - 3.2\mu) - z^{-4}1.1\mu .$$

After equating coefficients of z^{-1} these give;

$$b_2^2 = 0.063\mu$$

$$b_2^1 = 0.063 - 0.187\mu$$

$$b_2^o = 0.187\mu - 0.187$$

$$b_2^o = 3.1 - \mu .$$

From which,

$$\mu = 2.77,$$

and

$$b_2^o = 0.33$$

$$b_2^1 = -0.39$$

$$b_2^2 = 0.1745 .$$

Then,

$$l_{22} = (1 - z^{-1})(1 - 1.05z^{-1})^2(1 + 2.77z^{-1})$$

APPENDIX III . DERIVATION OF BASIC EQUATIONS

O is the zero matrix, I the identity matrix. Referring to Fig. 1: with U = O,

$$C = GD_1E_1$$

$$E_1 = R - D_2C$$

therefore,

$$C = GD_1\left[R - D_2C\right]$$

$$= GD_1R - GD_1D_2C$$

$$C + GD_1D_2C = GD_1R$$

$$\left[I + GD_1D_2\right]C = GD_1R$$

$$C = \left[I + GD_1D_2\right]^{-1} GD_1R.$$

With R = O,

$$C' = GD_1E_1' + U$$

$$E_1' = -D_2C'$$

then,

$$C' = -GD_1D_2C' + U$$

$$\left[I + GD_1D_2\right]C' = U$$

$$C' = \left[I + GD_1D_2\right]^{-1} U$$

From Equation (1),

$$C = KR$$

also $$C = GE_2$$

therefore,

$$GE_2 = KR$$

$$E_2 = G^{-1}KR.$$

Equation (8) gives,

$$L = \left[I + GD_1D_2\right]^{-1}$$

$$L\left[I + GD_1D_2\right] = I$$

$$L + LGD_1D_2 = I$$

$$LGD_1D_2 = I - L$$

$$LGD_1 = \left[I - L\right]D_2^{-1}$$

from Equation (9),

$$K = \left[I - L\right]D_2^{-1}$$

therefore,

$$L = I - KD_2.$$

Again from Fig. 1,

$$E'_2 = -D_1D_2C'$$

so

$$LGE'_2 = -LGD_1D_2C'$$

$$= -KD_2C'$$

$$E'_2 = -G^{-1}L^{-1}KD_2LU$$

but since KD_2 and L are diagonal matrices,

$$E'_2 = -G^{-1}KD_2U.$$